

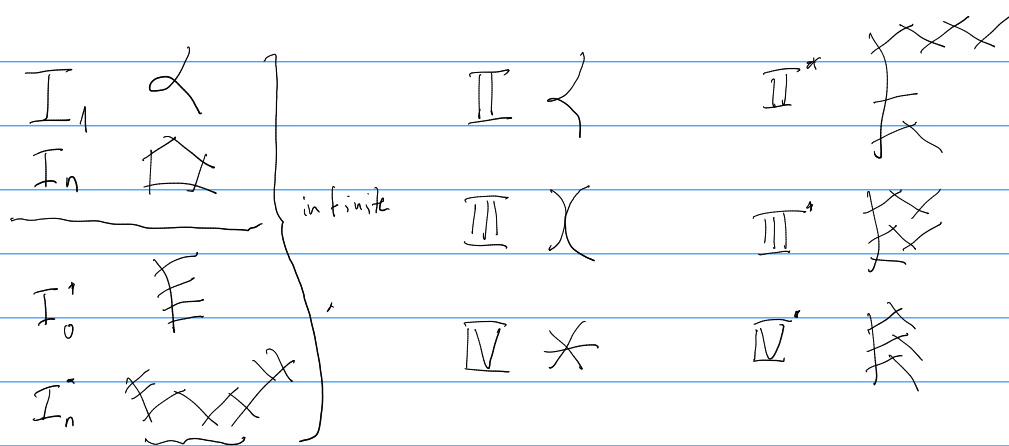
# Stacky Framework for the classification of singular fibers in fibered surfaces / $\mathbb{C}$ , char $k \gg g$

Setup:  $\pi: X \rightarrow B$  is a smooth surface fibered in curves of genus  $g$ .

Q: Can we classify the possible singular fibers?

We can assume: no  $(-1)$ -curves on fibers

Thm (Kodaira, Neen 60's) in genus 1: (assuming there's a section) the fibers are:



Thm (Ogg, Namikawa-Ueno 60's-70's): There's finitely many families of singular fibers in genus 2. (classified)

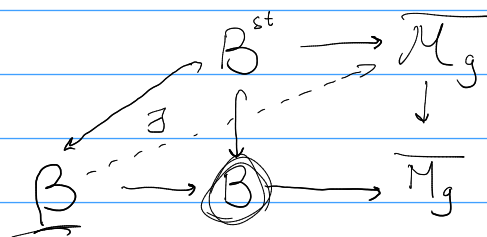
Thm (Artin-Winters '71): There's always finitely many families for any genus.

Thm (Winters '74): Any combinatorially admissible singular fiber appears as the central fiber of some fibration.

This is not effective. Also doesn't guarantee the general fiber is connected. (\*)

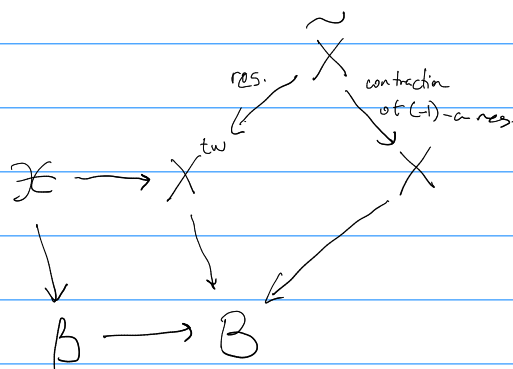
Thm/Algorithm (Dokchitser '25): There is an algorithm that classifies singular fibers in every genus. (\*) still holds.

Another approach to the classification.  $X \rightarrow B$   
 if  $B^{st} \subseteq B$  is the locus where the fibers are stable,



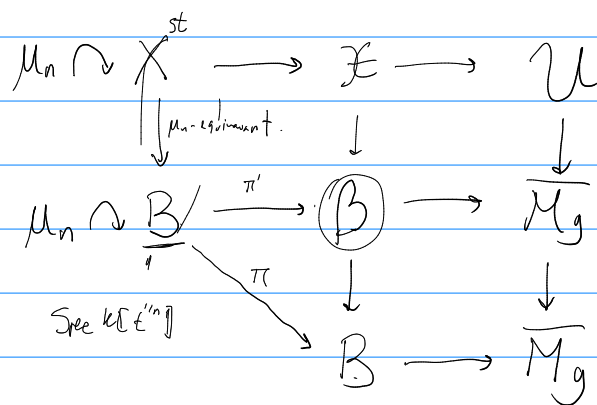
Thm (Abramovich-Vistoli '00's, Bresciani-Vistoli '20s): There is a moduli stack  $\mathcal{B} \rightarrow B$  with an extension to  $\overline{M}_g$

If we pull back the universal family  $\mathcal{U} \rightarrow \overline{M}_g$ : obtain  $\mathcal{X} \rightarrow \mathcal{B}$  where every fiber is stable.



If we assume that  $B = \text{Spec } k[t]$   $\mathcal{B} = [\text{Spec } k[t^{1/n}] / \mu_n]$

$$\zeta_n: t^{1/n} \mapsto \zeta_n t^{1/n}$$



$$\underline{X^{tw} = X^{st} / \mu_n}$$

if  $C \subseteq X^{st}$  is the stable central fiber,  $\mu_n$  must act <sup>as  $\langle \zeta_n \rangle$</sup>  as some automorphism  $f \in \text{Aut}(C)$ .

- The central fiber of  $X^{tw}$  is  $C/f$  with same multiplicities
- if  $p \in C$  on which  $f$  acts freely, then  $\bar{p} \in X^{tw}$  is smooth.
- If  $\underline{p} \in C$  is smooth, but fixed by some  $f^d$ , then

$$T_p X^{st} = \underline{T}_p B \oplus T_p C \quad \text{equivariant decomposition.}$$

$f^d$  acts as multiplication by some  $\zeta_{n/d}^a$  on  $T_p C$ , it also acts  $\zeta_{n/d}$  on  $T_p B$ .

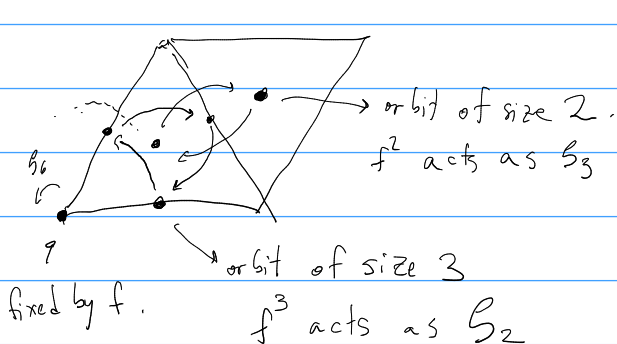
$$\begin{pmatrix} \zeta_{n/d} & 0 \\ 0 & \zeta_{n/d}^a \end{pmatrix}$$

$\hookrightarrow \bar{p} \in X^{tw}$  is a cyclic quotient singularity of type  $\frac{1}{n/d}(1, a)$ .

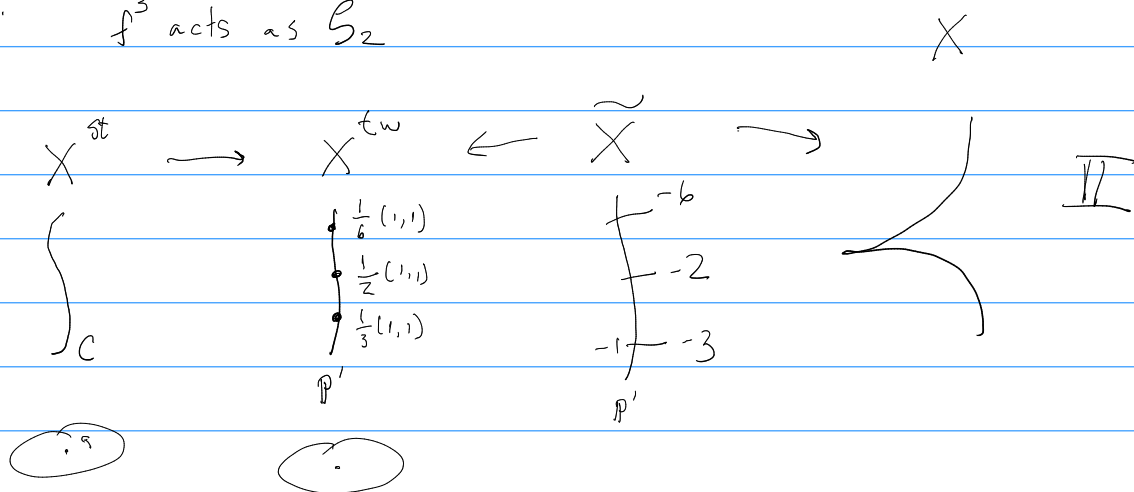
- If  $p \in C$  is a node,  $\bar{p} \in X^{tw}$  depends still on the action of  $f^d$  at  $T_p C$ .

Upshot: Pairs  $(C, f)$ , where  $C$  is stable,  $f \in \text{Aut}(C)$  (up to conjugacy) classify the families of singular fibers

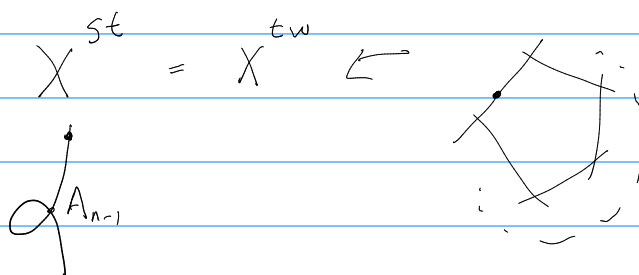
Eg:  $C = \text{elliptic curve}$   $j(C) = 0$   $\text{Aut}(C, p) = \mathbb{Z}/6\mathbb{Z}$ ,  $f \in \text{Aut}(C, p)$   
 correspond to complex multiplication by  $\zeta_6$



$$C/f = \mathbb{P}^1.$$

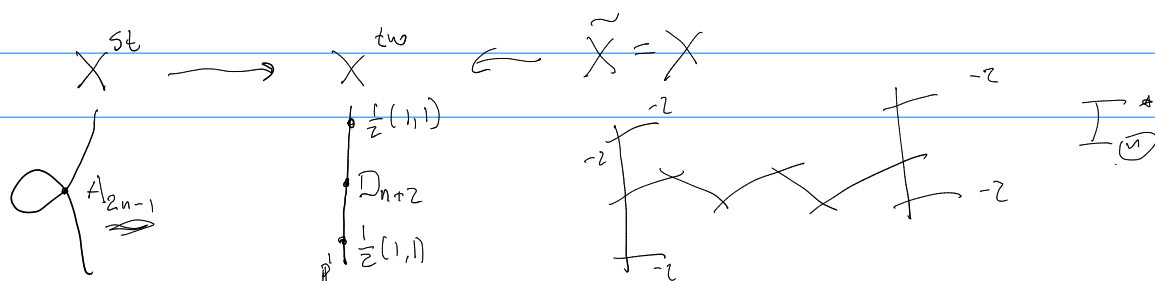


In: If I contract all  $(-2)$ -curves not passing through the section: this is already stable.



$\underline{I_0^*}$ :  $C = \mathcal{O}$ ,  $\text{Aut}(C, p) = \text{Aut}(G_m, p) = \mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$

$\sigma$  fixes the node, fixes  $p$ , fixes another point.



Hyper elliptic case : Reduces to classifying stable hyper elliptic curves  $C$ ,  $f \in \text{Aut}(C)$ .

When  $C$  is smooth:  $\exists! \overset{4}{C} \xrightarrow{2:1} \mathbb{P}^1 (\omega_C)$ .

$$1 \rightarrow \langle \sigma \rangle \rightarrow \text{Aut}(C) \xrightarrow{\psi_a} \text{Aut}(p^1, B) \rightarrow 1.$$

$B$  is the branch locus of  $C \rightarrow \mathbb{P}^1$ . Moreover: we can understand the action of  $f \in \text{Aut}(C)$  on tangent spaces by looking at the actions on tangent spaces of  $\varphi_* f$ .

When  $C \in \overline{\mathcal{H}}_g$  which is a limit of hyperelliptic curves: still is true that there is a unique generically 2-1 map

$$C \rightarrow P$$

where  $P$  is a nodal glos  $O$  curve.

$$1 \rightarrow (\mathbb{Z}/2\mathbb{Z})^N \rightarrow \text{Aut}(C) \rightarrow \text{Aut}(P, B) \rightarrow 1$$

$(P, \frac{1}{2}B) \in \overline{M}_{0, (2g+2) \times \frac{1}{2}}$ . Conversely, if  $(P, \frac{1}{2}B)$  is as before there exists a hyperelliptic curve  $C \rightarrow P$ .

Thm (WIP) There are finitely many hyp. "building blocks" from which we can construct all hyp. sing. fibers; by "glueing"

$$\begin{array}{ccc} \text{I} & \dots & \text{IV} \quad \text{V} \\ \text{I}^+ & \dots & \text{IV}^+ \quad \text{V}^+ \end{array} \quad \begin{array}{ccccc} M^1, & \bar{M}^1 & M^2 & \bar{M}^2 & \bar{M}^2 \\ \hline & & & & \downarrow \end{array}$$

generalizing  
 $2I_0$

$\overline{V}$     $\overline{III}$   
 $\overline{V}^{\perp}$     $\overline{III}^{\perp}$    agree on  
gens.